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Is Active Gravitational Mass Equal to Inertial Mass?

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It is possible, within the framework of general relativity, to define an active gravitational mass density of incoherent matter. It is not equal to the inertial mass density, except when at rest. The concept can be specialized to a single massive particle; again, its active gravitational mass is not equal to its inertial mass, except when it rests. A measurement of the impulse imparted to a test particle by a massive body passing nearby can establish the difference, and it may be possible to carry out this measurement in a laboratory. As a by-product of our computations we obtain a generalization to nonradial motion of the slowing-down effect in a Schwarzschild field.

The coupling constant κ in equation (3) between the gravitational field and the energy–momentum density field in general relativity is fixed by means of a correspondence principle with Newton's theory, applied to systems of incoherent matter in the limit of weak fields and low velocities. In particular, one uses the equality between the active gravitational and inertial masses. It turns out that κ is equal to Newton's gravitation constant. Once κ has been fixed, one may of course ask questions about conditions that are far removed from this limit. A natural question in this connection, which is the subject of this paper, is whether this equality between the masses remains true at high velocities. This question is connected with the mass–current effect predicted by general relativity. The best-known aspect of this effect is in the Lense–Thirring (1918) treatment of a rotating body. Based on this treatment is a

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measurable (in principle) correction term in the famous gyroscope precession equation (Schiff, 1967, 1960). Here we bring forth a different aspect of the mass–current effect. But (a big "but") whether it can really be discovered nowadays in a laboratory should be judged by an interested experimentalist.

Consider a four-dimensional differentiable manifold equipped with a projective structure. In every coordinate system x^i ($i = 0, 1, 2, 3$) one may describe most of the paths (we refer to them later as particles) by expressing their "locations" x^α ($\alpha = 1, 2, 3$) as functions of the "time" x^0. Then one may define the (observed) velocities by $v^\alpha \equiv (dx^\alpha/dx^0)$ and the (observed) accelerations by $a^\alpha \equiv (d^2x^\alpha/dx^{02})$ (concepts which depend on the choice of coordinates, but retain, at least partially, the expected physical meaning in coordinates provided with due physical essence). We set $c = 1$. Let Γ^i_{jk} be (locally) any symmetric affine connection such that its set of geodesics coincides with the projective structure. (Γ^i_{jk} is determined up to a projective transformation.) It is easy to show that

$$a^\alpha = -\Gamma^\alpha_{00} + (\Gamma^0_{00}\delta_\beta{}^\alpha - 2\Gamma^\alpha_{\beta 0})v^\beta + (2\Gamma^0_{\beta 0}\delta_\gamma{}^\alpha - \Gamma^\alpha_{\beta\gamma})v^\beta v^\gamma + \Gamma^0_{\beta\gamma}v^\beta v^\gamma v^\alpha \quad (1)$$

exactly. $A_\alpha \equiv -\Gamma^\alpha_{00}$ is the acceleration field of particles at rest.

We consider now a freely falling observer in the frame of general relativity. Suppose that to get a Newtonian notion of the gravitational field in his vicinity, he measures the gravitational accelerations A_α of slow (relative to him) test particles in Fermi coordinates. We need even less, that is

$$g_{ij} = \eta_{ij}, \qquad \Gamma^i_{jk} \equiv \left\{ {i \atop jk} \right\} = 0 \quad (2)$$

along $(x^0, 0, 0, 0)$.

Let us assume Einstein's equations

$$G^{ij} = -8\pi\kappa T^{ij} \quad (3)$$

By equations (1)–(3) we can find (for test particles at rest) at the spatial origin

$$A_{\alpha,\beta} - A_{\beta,\alpha} = 0, \qquad A_{\alpha,\alpha} = -R_{00} = 8\pi\kappa(T_{00} - \tfrac{1}{2}T_i^i) \quad (4)$$

This result can also be obtained from the geodesic deviation equation. If the source of the gravitational field is incoherent matter, $T^{ij} = \rho U^i U^j$, one finds

$$A_{\alpha,\beta} - A_{\beta,\alpha} = 0, \qquad A_{\alpha,\alpha} = -4\pi\kappa\rho(1 + v^2)(1 - v^2)^{-1} \quad (5)$$

where v is the velocity of the incoherent matter at the spatial origin: $U^i = (1 - v^2)^{-1/2}(1, \mathbf{v})$.

Comparison of (5) with Newton's gravitational field equations (even in a freely falling Euclidean nonrotating frame) shows that it is natural to *define* the (relative) gravitational active mass density of incoherent matter with respect to an observer moving with velocity v relative to the matter to be

$$\rho_A = \rho(1 + v^2)(1 - v^2)^{-1} \quad (6)$$

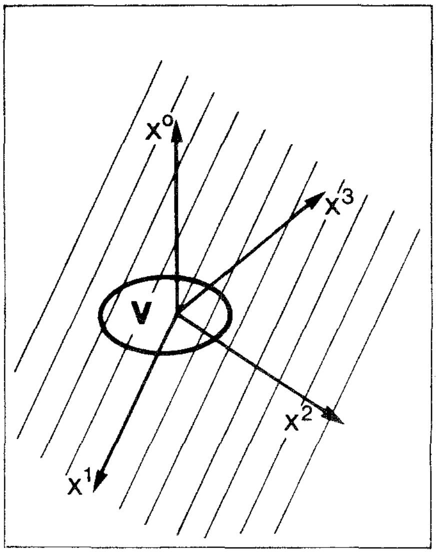

Fig. 1. Incoherent matter in Fermi coordinates. World lines of the particles are drawn. Distances between particles are much larger than their Schwarzschild radii.

This quantity represents locally the source of the gravitational acceleration field of particles at rest. We find it surprising that it is different from the (relative) inertial mass density [= energy density = $T^{00} = \rho(1 - v^2)^{-1}$]. Is this known? Is it possible to check this experimentally?

Interpreting incoherent matter as a mathematical limit to systems of numerous particles[2] with no interaction other than gravitational, we obtain from (5) the following flux of A_α through the boundary S of a small three-volume V around the spatial origin (Fig. 1):

$$\int_S \mathbf{A}\cdot \mathbf{dS} \simeq -4\pi\kappa V\rho(1 + v^2)(1 - v^2)^{-1}$$

The number of particles in V (world lines intersecting V) is $\rho m_0^{-1}V(1 - v^2)^{-1/2}$, if we assume that they all have the same proper mass m_0; it is easy to generalize to a mixture. Dividing the flux by $-4\pi\kappa$ times the number, we find the active gravitational mass of a particle of the incoherent matter;

$$m_A = m_0(1 + v^2)(1 - v^2)^{-1/2} \tag{7}$$

We now use a heuristic argument to establish the general conditions under which one can apply equations (5) [or even equations (4)] to finite

[2] Note that for given volume V and total proper mass M_0, $m_0/d \to 0$ as $N \to \infty$, where N is the number of particles, $M_0 = Nm_0$ and d is the average distance between close neighbors. $\kappa m_0/d \sim 0$ for "physical" incoherent matter, and vanishes for the mathematical abstraction.

domains or even globally, not just along a line. We then propose a laboratory experiment with some predicted results. Then we shall establish these predictions rigorously and try to get a deeper understanding of equations (5).

We think that equations (5) [or (4)] can be taken seriously globally when the following approximation is justified: (a) quasi stationarity, (b) linearity of equations in the gravitational field: weak field, (c) linearity of equations in mass: the gravitational energy is negligible with respect to kinetic energy (unbounded systems for which there is no "virial theorem").

Condition (a) is motivated by the fact that the solution A_α of equations (5) is determined by ρ at the instantaneous time rather than at the retarded time, as one should expect according to relativity theory. Conditions (b) and (c) are motivated by the linearity of equations (5) in A_α and in ρ.

It seems that one of the most suitable physical systems satisfying these conditions is a beam of particles, in which the distances between the particles are large compared to their Schwarzschild radii and dimensions. The following results are based on the interpretation of incoherent matter illustrated in Figure 1.

One can measure directly the gravitational force produced by a beam of known particles (acting on a test particle at rest) and compare it with the standard solution of equations (5). This prediction is different from that of Newton's theory and from that of the attractive assumption that the active gravitational mass is equal to the energy (= the inertial mass). Moreover, the beam–velocity dependence of the force is different according to these three alternatives. Equivalently, one may measure the impulse transferred to a test particle at rest from a body passing nearby, but still at a distance large compared with its Schwarzschild radius. This impulse is proportional to the product of the active gravitational mass m_A of the body and the time of interaction, and thus to $m_A v^{-1}$. On the Newtonian theory ($m_A = m_0$) we have $m_0 v^{-1}$. If we assume $m_A = m_I = m_0(1 - v^2)^{-1/2}$, we have $m_0 v^{-1}(1 - v^2)^{-1/2}$. According to our result, equation (7), we should have

$$m_0 v^{-1}(1 - v^2)^{-1/2}(1 + v^2)$$

For low velocities these impulses are, respectively, proportional to

$$m_0 v^{-1}, \qquad m_0 v^{-1}(1 + \tfrac{1}{2}v^2), \qquad m_0 v^{-1}(1 + \tfrac{3}{2}v^2)$$

An experiment designed to discover relativistic effects should be able to distinguish between $\frac{1}{2}v^2$ and $\frac{3}{2}v^2$. Also, the foregoing impulses are minimal at $v^2 = \infty, \frac{1}{2}, \frac{1}{3}$, respectively.

The latter, general relativistic, result $v^2 = \frac{1}{3}$ is also a critical value from another point of view: according to Carmeli (1972), there is a slowing-down effect for test particles projected radially in a Schwarzschild field which begins when the velocity (at infinity) is $1/\sqrt{3}$. We have not found an obvious con-

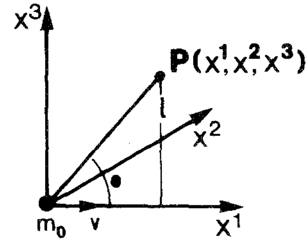

Fig. 2. Test particle at P at distance l from the line of motion x^1 of the body:
$x^0 = 0, \quad x^1 = r\cos\theta,$
$x^2 = r\sin\theta\cos\phi, \quad x^3 = r\sin\theta\sin\phi.$

nection between our result (after transforming to the body's rest frame) and Carmeli's; his result is restricted to radial motion and seems unsuitable for a laboratory experiment (it may be useful in an astronomical test). Later on we try to show that the results are really independent (Footnote 3).

We remember that our proposed experiment was based on some heuristic considerations. We shall now *prove* its predicted result [impulse proportional to $m_A(v)v^{-1}$, m_A given by equation (7)] for a spherically symmetric body. The only approximation we shall make is to consider only the r^{-2} part of the field A_α since we assume the test particle to be at a distance from the body that is large compared to the Schwarzschild radius $2\kappa m_0$ of the body.

The Schwarzschild metric in isotropic coordinates is

$$ds^2 = \frac{(1 - \kappa m_0/2r)^2}{(1 + \kappa m_0/2r)^2}\,dx^{0^2} - \left(1 + \frac{\kappa m_0}{2r}\right)^4 (dx^{1^2} + dx^{2^2} + dx^{3^2})$$

where $r \equiv (x^{1^2} + x^{2^2} + x^{3^2})^{1/2}$. We apply a Lorentz transformation

$$\{x^{0\prime} = (1 - v^2)^{-1/2}(x^0 + vx^1),\ x^{1\prime} = (1 - v^2)^{-1/2}(x^1 + vx^0),\ x^{2\prime} = x^2,\ x^{3\prime} = x^3\}$$

omit the primes, and calculate $A_\alpha = -\Gamma^\alpha_{00}$ at $x^0 = 0$, at which time the body is at the origin. Then we obtain (see Fig. 2)

$$A_1 = -\frac{\kappa m_0}{r^2}\,[(1 - 3v^2)(1 - v^2)^{-1}\alpha(\theta)\operatorname{ctg}\theta + O(\kappa m_0/r)]$$

$$A_2 = -\frac{\kappa m_0}{r^2}\,[(1 + v^2)\alpha(\theta)\cos\phi + O(\kappa m_0/r)]$$

$$A_3 = -\frac{\kappa m_0}{r^2}\,[(1 + v^2)\alpha(\theta)\sin\phi + O(\kappa m_0/r)]$$

where $\alpha(\theta) \equiv (1 - v^2)^{-1}[\sin^2 \theta + (1 - v^2)^{-1} \cos^2 \theta]^{-3/2} \sin \theta$. A_α is the force on a unit mass test particle at rest at the point **P** in Figure 2. Note that A_1 changes sign at $v^2 = \frac{1}{3}$; this is the generalization (to lowest order in r^{-1}) of Carmeli's slowing-down effect (1972) to nonradial motion. From symmetry the impulse $\int_{-\infty}^{\infty} A_\alpha \, dx^0$), exerted on the test particle during the whole motion of the source particle, is in the direction of the normal to the x^1 axis,[3] and its value (neglecting the $O(r^{-3})$ terms) is[4]

$$I = \kappa m_0(1 + v^2) \int_{-\infty}^{\infty} \alpha(\theta) r^{-2} v^{-1} \, dx^1 = 2\kappa \frac{m_0(1 + v^2)}{(1 - v^2)^{1/2} l} \cdot \frac{1}{v}$$

in accordance with our prediction. The Newtonian result is $I = 2\kappa m_0 l^{-1} v^{-1}$. In this sense, that of the total impulse exerted on a test particle of unit mass at rest, we may attribute to every spherical body (and not only to those that constitute incoherent matter) an active gravitational mass

$$m_A = m_0(1 + v^2)(1 - v^2)^{-1/2}$$

as in equation (7).

For a stationary beam of particles along the x^1 axis one obtains by superposition [neglecting the $O(r^{-3})$ term and the gravitational interaction between the particles in the beam, according to our assumptions] that A_α is directed normally toward the beam and that its value is

$$2\kappa n m_0(1 + v^2)(1 - v^2)^{-1/2} l^{-1}$$

where l is the distance from the beam and n is the number of particles (of rest mass m_0) per unit length of the beam. This is again in accordance with our predictions. A_α is a conservative field and satisfies along the beam Gauss' theorem with the correct source. To some extent these remarks explain equations (5), since we may refer to incoherent matter as a superposition of nonintersecting beams and, according to Footnote 2, incoherent matter (in which $\kappa m_0/d \sim 0$) particle feels just the "leading terms" of the A_α created by the other particles.

We conclude with some further remarks. We attributed to every particle a new concept: the (relative) active gravitational mass

$$m_A = m_0(1 + v^2)(1 - v^2)^{-1/2}$$

This is not an independent covariant quantity. The only covariant quantities describing a spinless particle are the scalar m_0 and the four-velocity U^i. But in

[3] It is clear now that Carmeli's result is independent of ours. With respect to the relative velocity, the first effect is "parallel" while the second is "normal."

[4] Substituting of $dx^0 = dx^1/v$, $x^1 = l \operatorname{ctg} \theta$, $r = l/\sin \theta$, in accordance with Figure 2, leads to an immediate integral.

analogy to $m = m_0(1 - v^2)^{-1/2}$, m_A expresses a definite physical property related to the particle's ability to create a gravitational field. The exact physical meaning was given (measure the impulse). We also dealt with the concept of active gravitational mass density, but we do not think that it is fruitful to speak of the active gravitational mass of big systems (apart from exceptional cases), since active gravitational mass, in contrast with energy, appears (even in Newtonian terms) only in expressions of the form $\sum (m_A/r)$ [rather than $\sum m_A$]. Based on (4), it is natural, however, to generalize our definition (6) of active gravitational mass density to arbitrary systems, in the form $\rho_A = -T_{00} + \frac{1}{2}T_i^i$. With this terminology the strong energy condition (Hawking and Ellis, 1973) is equivalent to the claim that the active gravitational mass density is nonnegative. Finally this discussion places a question mark on the validity of the commonly accepted belief that the inertial, passive gravitational and active gravitational masses are all equal to each other and (up to c^2) to the energy.

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